

# CONICAL UPPER DENSITY THEOREMS AND POROSITY OF MEASURES

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ABSTRACT. We study how measures with finite lower density are distributed around  $(n - m)$ -planes in small balls in  $\mathbb{R}^n$ . We also discuss relations between conical upper density theorems and porosity. Our results may be applied to a large collection of Hausdorff and packing type measures.

## 1. INTRODUCTION

Conical density theorems are used in geometric measure theory to derive geometric information from given metric information. Classically, they deal with the distribution of the  $s$ -dimensional Hausdorff measure,  $\mathcal{H}^s$ . The main applications of conical density theorems concern rectifiability, see [14], but they have been applied also elsewhere in geometric measure theory, for example, in the study of porous sets, see [13] and [11]. The upper conical density results, going back to Besicovitch [2] and Marstrand [12], show that under certain conditions there is a lot of  $A$  near each  $k$ -dimensional linear subspace of  $\mathbb{R}^n$  in some small balls  $B(x, r)$ . Besides Besicovitch and Marstrand, the theory of upper conical density theorems has been developed by Morse and Randolph [15], Federer [7], and Salli [16]. For a partial survey on various conical density theorems for measures on  $\mathbb{R}^n$ , consult [17]. A sample result is the following (Salli [16, Theorem 3.1]): If  $V \in G(n, n - m)$ , where  $G(n, n - m)$  denotes the space of all  $(n - m)$ -dimensional linear subspaces of  $\mathbb{R}^n$ ,  $0 < \alpha < 1$ ,  $A \subset \mathbb{R}^n$ ,  $0 < \mathcal{H}^s(A) < \infty$ , and  $s > m \geq 1$ , then

$$\limsup_{r \downarrow 0} \frac{\mathcal{H}^s(A \cap X(x, r, V, \alpha))}{(2r)^s} \geq c \quad (1.1)$$

for  $\mathcal{H}^s$ -almost all  $x \in A$ , where  $c > 0$  is a constant depending only on  $n, m, s$ , and  $\alpha$ . Here

$$X(x, V, r, \alpha) = \{y \in B(x, r) : \text{dist}(y - x, V) < \alpha|y - x|\},$$

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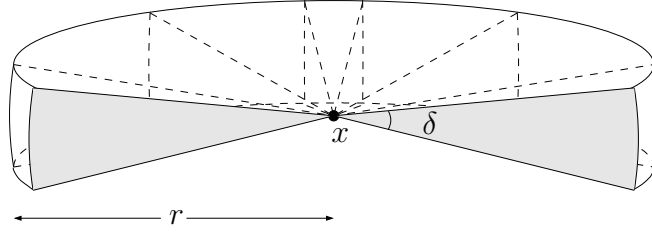


FIGURE 1. The set  $X(x, r, V, \alpha) \setminus H(x, \theta, \eta)$  when  $n = 3$ ,  $m = 1$ ,  $\alpha = \sin(\delta/2)$ , and  $\theta$  is pointing up from the paper.

where  $B(x, r) \subset \mathbb{R}^n$  is the closed ball with center at  $x$  and radius  $r > 0$ . Open balls are denoted by  $U(x, r)$ . Clearly, (1.1) is not true anymore if  $s \leq m$  since in this case it might happen that  $A \subset V^\perp$ .

In [13], Mattila improved the above result by showing that it is not necessary to fix  $V$  in (1.1). More precisely, he proved that if  $A \subset \mathbb{R}^n$ ,  $0 < \mathcal{H}^s(A) < \infty$ ,  $s > m$ , and  $0 < \alpha < 1$ , then for a constant  $c > 0$  depending only on  $n$ ,  $m$ ,  $s$ , and  $\alpha$ ,

$$\limsup_{r \downarrow 0} \inf_C \frac{\mathcal{H}^s(A \cap B(x, r) \cap C_x)}{(2r)^s} \geq c \quad (1.2)$$

for  $\mathcal{H}^s$ -almost all  $x \in A$ , where  $C_x = \{x\} + \bigcup C$  and the infimum is taken over all Borel sets  $C \subset G(n, n-m)$  for which  $\gamma_{n, n-m}(C) > \alpha$ . Here  $\gamma_{n, n-m}$  denotes the unique Borel regular probability measure on  $G(n, n-m)$  invariant under the orthogonal group  $O(n)$ , see [14, §3.9]. As an immediate corollary to Mattila's result, under the same assumptions as in (1.1), we have

$$\limsup_{r \downarrow 0} \inf_{V \in G(n, n-m)} \frac{\mathcal{H}^s(A \cap X(x, r, V, \alpha))}{(2r)^s} \geq c \quad (1.3)$$

for  $\mathcal{H}^s$ -almost all  $x \in A$ , where  $c > 0$  depends only on  $n$ ,  $m$ ,  $s$ , and  $\alpha$ , see [14, §11]. Although the constant in (1.1) is much better than that of (1.3), still (1.3) is a significant improvement of (1.1): It shows that in the sense of the measure  $\mathcal{H}^s$ , there are arbitrarily small scales such that almost all points of  $A$  are well surrounded by  $A$ .

In what follows, we shall also allow  $m = 0$ , in which case  $G(n, n-m) = G(n, n) = \{\mathbb{R}^n\}$  and  $X(x, r, \mathbb{R}^n, \alpha) = B(x, r)$ . If  $\mu$  is a measure on  $\mathbb{R}^n$  and  $A \subset \mathbb{R}^n$ , we use the notation  $\mu|_A$  for the restriction measure, that is  $\mu|_A(B) = \mu(A \cap B)$  for  $B \subset \mathbb{R}^n$ .

The proof of (1.2) is nontrivial and it is based on Fubini-type arguments and an elegant use of the so-called sliced measures. Since the geometry of the cones  $X(x, r, V, \alpha)$  is simpler than that of the cones  $C_x$  in (1.2), it is natural to ask for an elementary proof of (1.3). In [11], such a proof was given and the technique used there does not require the cones to be symmetric. Namely, given  $s > m$ ,

$0 < \alpha < 1$ ,  $0 < \eta < 1$ , and  $A \subset \mathbb{R}^n$  with  $0 < \mathcal{H}^s(A) < \infty$ , it was shown in [11, Theorem 2.5] that there is a constant  $c > 0$  depending only on  $n, m, s, \alpha$ , and  $\eta$  so that

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mathcal{H}^s(A \cap X(x, r, V, \alpha) \setminus H(x, \theta, \eta))}{(2r)^s} \geq c \quad (1.4)$$

for  $\mathcal{H}^s$ -almost all  $x \in A$ . Here  $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$  and

$$H(x, \theta, \eta) = \{y \in \mathbb{R}^n : (y - x) \cdot \theta > \eta|y - x|\}$$

is the almost half-space centered at  $x$  pointing to the direction of  $\theta$  with the opening angle  $0 < \beta < \pi$  given by  $\cos(\beta/2) = \eta$ .

At first glance, the cones  $X(x, r, V, \alpha) \setminus H(x, \theta, \eta)$  may seem a bit artificial. Let us look at some special cases. To help the geometrical visualization, it might be helpful to take  $\alpha$  and  $\eta$  close to 0 and  $\theta \in V \cap S^{n-1}$ , see Figure 1. When  $m = n - 1$ , the claim (1.4) is equivalent to

$$\limsup_{r \downarrow 0} \inf_{\varrho \in S^{n-1}} \frac{\mathcal{H}^s(A \cap X^+(x, r, \varrho, \alpha))}{(2r)^s} \geq c(n, s, \alpha) > 0, \quad (1.5)$$

where

$$\begin{aligned} X^+(x, r, \varrho, \alpha) &= \{y \in B(x, r) : (y - x) \cdot \varrho > (1 - \alpha^2)^{1/2}|y - x|\} \\ &= B(x, r) \cap H(x, \varrho, (1 - \alpha^2)^{1/2}). \end{aligned}$$

Since  $X(x, r, V, \alpha) = X^+(x, r, \varrho, \alpha) \cup X^+(x, r, -\varrho, \alpha)$  whenever  $V = \{t\varrho : t \in \mathbb{R}\} \in G(n, 1)$ , we see from (1.5) that the cone  $X(x, r, V, \alpha)$  in (1.3) may be replaced by  $X^+(x, r, \varrho, \alpha)$  when  $m = n - 1$ . This case was also considered in Mattila [13].

When  $0 < m < n - 1$ , there is no more natural way to divide the cones  $X(x, r, V, \alpha)$  into two or more similar parts, and we are led to replace the cones  $X^+(x, r, \varrho, \alpha)$  by  $X(x, r, V, \alpha) \setminus H(x, \theta, \eta)$ . However, the main reason for considering the densities (1.4) in [11] comes from porosity. Mattila's result (1.5) implies that the lower porosity of the measure  $\mathcal{H}^s|_A$  can not be too close to the maximum value  $\frac{1}{2}$  when  $s > n - 1$ . This leads into a relatively sharp dimension estimate for lower porous sets with porosity close to  $\frac{1}{2}$ , see [13] and [14, §11]. In a similar manner, the result (1.4) leads to a dimension estimate for the so called  $k$ -porous sets, introduced in [11].

When  $m = 0$ , the statement (1.4) is applicable to all  $0 < s \leq n$  and reads

$$\limsup_{r \downarrow 0} \inf_{\theta \in S^{n-1}} \frac{\mathcal{H}^s(A \cap B(x, r) \setminus H(x, \theta, \eta))}{(2r)^s} \geq c(n, s, \eta) > 0, \quad (1.6)$$

thus showing that for almost all  $x \in A$  the set  $A$  (or the measure  $\mathcal{H}^s|_A$ ) can not be concentrated on almost half-balls  $B(x, r) \cap H(x, \theta, \eta)$  for all small scales.

Easy examples, such as  $A = S^1 \subset \mathbb{R}^2$ , show that one can not replace the almost half-spaces  $H(x, \theta, \eta)$  by the half-spaces  $H(x, \theta, 0)$  in (1.6).

The statement (1.4) as well as its more general formulation [11, Theorem 2.6] deals with measures having finite upper density with respect to some gauge function. In particular, they do not in general apply to packing type measures. Thus there is a need for upper conical density theorems concerning measures with finite lower density and (possibly) infinite upper density. In our main result, Theorem 2.4, we generalize the result (1.4) for measures with finite lower density with respect to an appropriate gauge. The main application of this generalization, Corollary 2.5, is a conical density theorem for the  $s$ -dimensional packing measure,  $\mathcal{P}^s$ . Our result may also be applied to a large collection of Hausdorff and packing type measures which are determined using a variety of gauges. Besides the generalizations of (1.1) given in [18], there seems to be no conical density theorems of a similar type in the literature for other than Hausdorff measures.

Theorem 2.4 may be viewed as a dual result to the known lower conical density theorems which tell roughly that under certain conditions, we may find, around typical points, some small half balls with almost no measure. See, for example, [18, Theorem 2.1].

In §3, we discuss connections between conical densities and porosity. Namely, we show how conical density theorems may be used to obtain upper bounds for the porosity of measures. We shall also discuss the sharpness of our main result using this connection. Finally, in §4 we pose some open problems.

We finish the introduction by setting down some notation. Throughout the paper, we assume that  $h$  is a positive function defined on some small interval  $(0, r_0)$ . We shall also assume, for simplicity, that  $h$  is nondecreasing though this is not essential. If  $\mu$  is a Borel measure on  $\mathbb{R}^n$  (i.e. an outer measure defined on all subsets of  $\mathbb{R}^n$  such that Borel sets are measurable) and  $x \in \mathbb{R}^n$ , the upper and lower  $\mu$ -densities at  $x$  with respect to  $h$  are given by

$$\begin{aligned}\overline{D}_h(\mu, x) &= \limsup_{r \downarrow 0} \frac{\mu(B(x, r))}{h(2r)}, \\ \underline{D}_h(\mu, x) &= \liminf_{r \downarrow 0} \frac{\mu(B(x, r))}{h(2r)}.\end{aligned}$$

If  $V \in G(n, m)$ ,  $x \in \mathbb{R}^n$ , and  $\lambda > 0$ , we define

$$V_x(\lambda) = \{y \in \mathbb{R}^n : \text{dist}(y - x, V) \leq \lambda\}.$$

## 2. CONICAL UPPER DENSITY THEOREMS

To prove our main result, Theorem 2.4, we need the following two geometrical lemmas. The first one is due to Erdős and Füredi [5], see also [11, Lemma 2.1].

**Lemma 2.1.** *For a given  $0 < \beta < \pi$ , there is  $q = q(n, \beta) \in \mathbb{N}$  such that in any set of  $q$  points in  $\mathbb{R}^n$ , there are always three points which determine an angle between  $\beta$  and  $\pi$ .*

For  $0 < \eta \leq 1$  we define  $t(\eta) = (\eta^2 + 4)^{1/2}/\eta$  and  $\gamma(\eta) = 1/t(\eta)$ . Notice that  $t(\eta) \geq 2$  and  $\eta/5^{1/2} \leq \gamma(\eta) \leq \eta/2$ . An easy calculation yields the following, see [11, Lemma 2.3].

**Lemma 2.2.** *Suppose  $y \in \mathbb{R}^n$ ,  $\theta \in S^{n-1}$ ,  $0 < \eta \leq 1$ ,  $t \geq t(\eta)$ , and  $\gamma = \gamma(\eta)$ . If  $z \in \mathbb{R}^n \setminus (B(y, tr) \cup H(y, \theta, \gamma))$ , then  $B(z, r) \cap H(y, \theta, \eta) = \emptyset$ .*

Below, we include one more simple lemma.

**Lemma 2.3.** *Let  $m \geq 0$  be an integer and  $h: (0, r_0) \rightarrow (0, \infty)$ . Then the following conditions are equivalent:*

- (1) *There is  $r_0 > 0$  such that*

$$\frac{h(\varepsilon r)}{\varepsilon^m h(r)} \xrightarrow{\varepsilon \downarrow 0} 0 \quad (2.1)$$

*uniformly for all  $0 < r < r_0$ .*

- (2) *There is  $s > m$  and  $r_0, \varepsilon_0 > 0$  such that*

$$h(\varepsilon r) \leq \varepsilon^s h(r) \quad (2.2)$$

*for all  $0 < r < r_0$  and  $0 < \varepsilon < \varepsilon_0$ .*

- (3) *There is  $0 < c < 1$  such that*

$$\limsup_{r \downarrow 0} \frac{h(cr)}{h(r)} < c^m.$$

*Proof.* By (1), there is  $0 < \delta < 1$  and  $0 < c < 1$  such that  $h(\delta r) < c\delta^m h(r)$  for all  $0 < r < r_0$ . Let  $s_0 > 0$  be such that  $\delta^{s_0} = c$  and take  $m < s < m + s_0$  and  $0 < \varepsilon_0 < \delta$  for which  $\varepsilon^{m+s_0} \leq \delta^{m+s_0} \varepsilon^s$  for all  $0 < \varepsilon < \varepsilon_0$ . Given  $0 < \varepsilon < \varepsilon_0$ , let  $k \in \mathbb{N}$  be such that  $\delta^{k+1} < \varepsilon \leq \delta^k$ . Then

$$\begin{aligned} h(\varepsilon r) &\leq h(\delta^k r) \leq c^k \delta^{km} h(r) = \delta^{k(m+s_0)} h(r) = \varepsilon^{m+s_0} (\delta^k / \varepsilon)^{m+s_0} h(r) \\ &\leq (\delta^{k+1} / \varepsilon)^{m+s_0} \varepsilon^s h(r) < \varepsilon^s h(r) \end{aligned}$$

for all  $0 < r < r_0$  giving (2). That (3) implies (1) follows by a similar reasoning. Finally, notice that (2) clearly implies (3).  $\square$

Next we prove our main result concerning the distribution of measures with finite lower density.

**Theorem 2.4.** *Let  $\alpha, \eta \in (0, 1)$  and suppose  $h: (0, r_0) \rightarrow (0, \infty)$  satisfies (2.1) for some  $m \in \{0, \dots, n-1\}$ . If  $\mu$  is a Borel measure on  $\mathbb{R}^n$  with  $\underline{D}_h(\mu, x) < \infty$*

for  $\mu$ -almost all  $x \in \mathbb{R}^n$  then

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mu(X(x, r, V, \alpha) \setminus H(x, \theta, \eta))}{h(2r)} \geq c \overline{D}_h(\mu, x) \quad (2.3)$$

for  $\mu$ -almost all  $x \in \mathbb{R}^n$ . Here  $c > 0$  is a constant depending only on  $n, m, \varepsilon_0, s, \alpha$  and  $\eta$  where  $\varepsilon_0 > 0$  and  $s > m$  are as in Lemma 2.3.

*Proof.* Let us first sketch the main idea of the proof: Suppose our theorem is false. Then there is a closed exceptional set  $F \subset \mathbb{R}^n$  with positive  $\mu$ -measure so that for all small scales  $r > 0$  and for all points  $x$  of  $F$ , there are  $\theta$  and  $V$  so that  $\mu(X(x, r, V, \alpha) \setminus H(x, \theta, \eta))$  is very small compared to  $h(2r)$ . A simple covering argument on  $G(n, n-m)$  implies that at each small ball  $B = B(z, r)$  centered in  $F$ , we may fix  $V \in G(n, n-m)$  so that the measure  $\mu(X(x, r, V, \alpha) \setminus H(x, \theta, \eta))$  is small for some  $\theta$  for a set of points  $x \in F \cap B$  whose measure is comparable to  $h(2r)$ . This implies that for  $\lambda > 0$ , we may find  $y \in F \cap B$  so that the measure in  $V_y(\lambda r)$  is comparable to  $\lambda^m h(2r)$ . But our assumption implies that if  $\lambda$  is small, then this measure is essentially contained in at most  $q-1$  balls of radius  $\lambda r$ , the number  $q$  being determined by Lemma 2.1. Thus, there is a ball  $B(w, \lambda r) \subset B$  so that  $\mu(F \cap B(w, \lambda r)) \approx \lambda^m h(2r)$ . Iterating this, we find a sequence of balls  $B_1 \supset B_2 \supset \dots$  so that  $\text{diam}(B_k) \approx \lambda^k$  and  $\mu(F \cap B_k) \approx \lambda^{mk}$ . By (2.1), this implies  $\underline{D}_h(\mu, x) = \infty$  for the point  $x$  given by  $\{x\} = \bigcap_k B_k$ . This gives a contradiction since we may choose  $F$  at the outset so that the lower density  $\underline{D}_h(\mu, x)$  is finite for all points of  $F$ .

We shall now verify in detail the steps described heuristically above. We assume that  $m \geq 1$ . The case  $m = 0$  is easier and is discussed at the end of the proof. We may assume that  $\mu$  is finite since  $\mu$ -almost all of  $\mathbb{R}^n$  is contained in a countable union of open balls, each of finite  $\mu$ -measure. This follows by a straightforward covering argument since  $\underline{D}_h(\mu, x) < \infty$  almost everywhere. Let  $\varepsilon_0 > 0$  and  $s > m$  be as in Lemma 2.3. We shall prove that for any finite collection,  $\{V^1, \dots, V^l\} \subset G(n, n-m)$ ,

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ i \in \{1, \dots, l\}}} \frac{\mu(X(x, r, V^i, \alpha) \setminus H(x, \theta, \eta))}{h(2r)} \geq c(n, m, s, \varepsilon_0, \eta, \alpha, l) \overline{D}_h(\mu, x)$$

for  $\mu$ -almost all  $x \in \mathbb{R}^n$  from which (2.3) follows by the compactness of  $G(n, n-m)$ , see [11, proof of Theorem 2.5] for details.

Set  $t = \max\{t(\eta), 1 + 3/\alpha\}$ ,  $\gamma = \gamma(\eta)$ , where  $t(\eta)$  and  $\gamma(\eta)$  are as in Lemma 2.2, and take  $\beta < \pi$  so that the opening angle of  $H(x, \theta, \gamma)$  is smaller than  $\beta$ . Let  $q = q(n, \beta)$  be as in Lemma 2.1. Moreover, define  $c_1 = 2^m m^{m/2}$ ,  $c_2 = 2^n n^{n/2}$ ,  $d = (3c_1 l(q-1))^{-1}$ ,  $\lambda = \min\{2^{-1} t^{s/(m-s)} d^{1/(s-m)}, \varepsilon_0/(3t)\}$ , and  $c = c(n, m, s, \eta, \alpha, l) = \lambda^n / (6c_1 c_2 \ell 3^s)$ . These definitions together with (2.2) guarantee the following three facts: If  $0 < r < r_0$ ,  $k \in \mathbb{N}$ ,  $V \in G(n, n-m)$ ,  $z \in \mathbb{R}^n$ , and  $x, y \in V_z(\lambda r)$  with

$|x - y| \geq t\lambda r$ , then

$$B(y, \lambda r) \subset X(x, V, \alpha), \quad (2.4)$$

$$h(6(t\lambda)^k r) < 3^s d^k \lambda^{km} h(2r), \quad (2.5)$$

$$d\lambda^{m-s} t^{-s} \geq 2^{s-m}. \quad (2.6)$$

We give some details for the convenience. The claim (2.4) follows since  $d(w - x, V) \leq 3\lambda r \leq \alpha(t - 1)\lambda r < \alpha|w - x|$  for all  $w \in B(y, \lambda r)$  by the definition of  $t$ . To prove (2.5), we use (2.2) to get  $h(6(t\lambda)^k r) \leq 3^s t^{ks} \lambda^{ks} h(2r)$ . The definition of  $\lambda$  easily gives  $t^{ks} \lambda^{ks} < d^k \lambda^{km}$ . Finally, the bound (2.6) comes directly from the definition of  $\lambda$ .

Let  $0 < M < \infty$  and define

$$A = \{x \in \mathbb{R}^n : \overline{D}_h(\mu, x) > M \text{ and } \underline{D}_h(\mu, x) < \infty\}.$$

The set  $A$  is Borel since  $x \mapsto \overline{D}_h(\mu, x)$  and  $x \mapsto \underline{D}_h(\mu, x)$  are Borel functions. It suffices to show that

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ i \in \{1, \dots, l\}}} \frac{\mu(X(x, r, V^i, \alpha) \setminus H(x, \theta, \eta))}{h(2r)} \geq cM$$

for almost all  $x \in A$ . Suppose to the contrary that there exists a set  $F \subset A$  with  $\mu(F) > 0$  and  $0 < r_1 < r_0$  such that for every  $x \in F$  and  $0 < r < r_1$ , there are  $i \in \{1, \dots, l\}$  and  $\theta \in S^{n-1}$  with

$$\mu(X(x, r, V^i, \alpha) \setminus H(x, \theta, \eta)) < cMh(2r). \quad (2.7)$$

Going into a subset, if necessary, we may assume that  $F$  is closed.

Choose  $x \in F$  such that  $\lim_{r \downarrow 0} \mu(F \cap B(x, r)) / \mu(B(x, r)) = 1$  and  $0 < r < r_1/3$  such that  $\mu(F \cap B(x, r)) \geq Mh(2r)$ . To simplify the notation, we assume that  $r = 1$  and  $h(2) = 1$ . We can do this by replacing  $\mu$  by  $\tilde{\mu}(A) = \mu(rA)/h(2r)$  and  $h$  by  $\tilde{h}(t) = h(rt)/h(2r)$ . Our aim is to find  $z \in F$  for which  $\underline{D}_h(\mu, z) = \infty$  and this is clearly equivalent to  $\underline{D}_{\tilde{h}}(\tilde{\mu}, z/r) = \infty$ .

Let  $B_0 = B(x, 1)$ . Suppose that  $B_k = B(x_k, (t\lambda)^k)$  has been defined for  $k \geq 0$  so that  $\mu(F \cap B_k) \geq Md^k \lambda^{mk}$ . Take  $x_{k+1} \in F \cap B_k$  which maximizes the function  $y \mapsto \mu(F \cap B(y, (t\lambda)^{k+1}))$  in  $F \cap B_k$ . There is such a point because  $F \cap B_k$  is compact and the function  $y \mapsto \mu(F \cap B(y, (t\lambda)^{k+1}))$  is upper semicontinuous on  $F \cap B_k$ . Define  $B_{k+1} = B(x_{k+1}, (t\lambda)^{k+1})$ . Our aim is to estimate the measure  $\mu(F \cap B_{k+1})$  from below. Define, for  $i \in \{1, \dots, l\}$ ,

$$\begin{aligned} \tilde{C}_i &= \{x \in F \cap B_k : \mu(X(x, 3(t\lambda)^k, V^i, \alpha) \setminus H(x, \theta, \eta)) \\ &\quad < cMh(6(t\lambda)^k) \text{ for some } \theta \in S^{n-1}\}. \end{aligned}$$

Fix  $i \in \{1, \dots, l\}$  for which  $\mu(\tilde{C}_i) \geq \mu(F \cap B_k)/l \geq Md^k \lambda^{mk}/l$  and take a compact  $C_i \subset \tilde{C}_i$  with  $\mu(C_i) > \mu(\tilde{C}_i)/2$ . We may cover the set  $V^{i\perp} \cap B_k$  with  $c_1 \lambda^{-m}$  balls

of radius  $t^k \lambda^{k+1}$  and hence there exists  $y \in V^{i\perp} \cap B_k$  for which

$$\mu(C_i \cap V_y^i(t^k \lambda^{k+1})) \geq 2^{-1} c_1^{-1} \ell^{-1} M d^k \lambda^{m(k+1)}. \quad (2.8)$$

Next we shall choose  $q$  points as follows: Choose a point  $y_1 \in C_i \cap V_y^i(t^k \lambda^{k+1})$  such that the ball  $B(y_1, t^k \lambda^{k+1})$  has largest  $\mu|_F$  measure among the balls centered at  $C_i \cap V_y^i(t^k \lambda^{k+1})$  with radius  $t^k \lambda^{k+1}$ . If  $y_1, \dots, y_p, p \in \{1, \dots, q-1\}$ , have already been chosen, we choose  $y_{p+1} \in C_i \cap V_y^i(t^k \lambda^{k+1}) \setminus \bigcup_{j=1}^p U(y_j, (t\lambda)^{k+1})$  so that the ball  $B(y_{p+1}, t^k \lambda^{k+1})$  has maximal  $\mu|_F$  measure among the balls centered at  $C_i \cap V_y^i(t^k \lambda^{k+1}) \setminus \bigcup_{j=1}^p U(y_j, (t\lambda)^{k+1})$  with radius  $t^k \lambda^{k+1}$ . If our process of selecting the points  $y_j$  terminates before the  $q$ :th step, i.e. the balls  $\bigcup_{j=1}^p U(y_j, (t\lambda)^{k+1})$  cover the set  $F \cap C_i \cap V_y^i(t^k \lambda^{k+1})$  for some  $p < q$ , we get

$$\begin{aligned} \sum_{j=1}^p \mu(F \cap B(y_j, (t\lambda)^{k+1})) &\geq \mu(C_i \cap V_y^i(t^k \lambda^{k+1})) \\ &\geq 2^{-1} c_1^{-1} \ell^{-1} M d^k \lambda^{m(k+1)} \end{aligned} \quad (2.9)$$

by (2.8).

Suppose now that the process did not terminate before the  $q$ :th step. Since the set  $V_y^i(t^k \lambda^{k+1}) \cap B_k$  may be covered by  $c_2 \lambda^{m-n}$  balls of radius  $t^k \lambda^{k+1}$ , using (2.8), we get

$$\begin{aligned} \mu(F \cap B(y_q, t^k \lambda^{k+1})) &\geq c_2^{-1} \lambda^{n-m} \left( 2^{-1} c_1^{-1} \ell^{-1} M d^k \lambda^{m(k+1)} \right. \\ &\quad \left. - \sum_{j=1}^{q-1} \mu(F \cap B(y_j, (t\lambda)^{k+1})) \right). \end{aligned} \quad (2.10)$$

According to Lemma 2.1, we may choose three points  $w, w_1, w_2$  from the set  $\{y_1, \dots, y_q\}$  such that for each  $\theta \in S^{n-1}$  there is  $j \in \{1, 2\}$  for which  $w_j \in \mathbb{R}^n \setminus (B(w, (t\lambda)^{k+1}) \cup H(w, \theta, \gamma))$ . We obtain, using Lemma 2.2, that for each  $\theta \in S^{n-1}$  there is  $j \in \{1, 2\}$  such that

$$B(w_j, t^k \lambda^{k+1}) \subset B(w, 3(t\lambda)^k) \setminus H(w, \theta, \eta)$$

and hence (2.4) implies that also

$$B(w_j, t^k \lambda^{k+1}) \subset X(w, 3(t\lambda)^k, V^i, \alpha) \setminus H(w, \theta, \eta), \quad (2.11)$$

see Figure 2. Since  $w \in C_i$  there is  $\theta \in S^{n-1}$  so that  $\mu(X(w, 3(t\lambda)^k, V^i, \alpha) \setminus H(w, \theta, \eta)) < c M h(6(t\lambda)^k)$ . Choosing  $j \in \{1, 2\}$  for which (2.11) holds, we get

$$\begin{aligned} \mu(F \cap B(y_q, t^k \lambda^{k+1})) &\leq \mu(F \cap B(w_j, t^k \lambda^{k+1})) \\ &\leq \mu(X(w, 3(t\lambda)^k, V^i, \alpha) \setminus H(w, \theta, \eta)) \\ &< c M h(6(t\lambda)^k). \end{aligned} \quad (2.12)$$



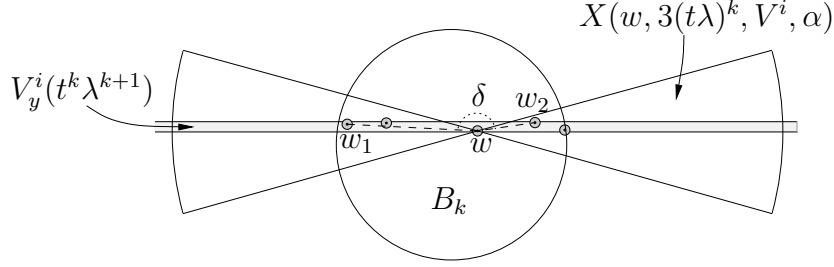


FIGURE 2. Illustration for the proof of Theorem 2.4. The angle  $\delta$  formed by the points  $w_1, w$ , and  $w_2$  is greater than  $\beta$ .

Consequently, using (2.10), (2.12), (2.5), and the definitions of  $c$ ,  $c_1$ ,  $c_2$ , and  $d$ , we get

$$\begin{aligned}
 \sum_{j=1}^{q-1} \mu(F \cap B(y_j, (t\lambda)^{k+1})) &> 2^{-1} c_1^{-1} \ell^{-1} M d^k \lambda^{m(k+1)} - c_2 c M h (6(t\lambda)^k) \lambda^{m-n} \\
 &> 2^{-1} c_1^{-1} \ell^{-1} M d^k \lambda^{m(k+1)} - c_2 c M 3^s d^k \lambda^{m(k+1)} \lambda^{-n} \\
 &= 3^{-1} c_1^{-1} \ell^{-1} M d^k \lambda^{m(k+1)} \\
 &= (q-1) M d^{k+1} \lambda^{m(k+1)}.
 \end{aligned}$$

It follows that there is  $y_j \in \{y_1, \dots, y_{q-1}\}$  for which  $\mu(F \cap B(y_j, (t\lambda)^{k+1})) \geq M(d\lambda^m)^{k+1}$ . Inspecting the above calculation, we see that this is true also if (2.9) holds. Thus we get

$$\mu(F \cap B_{k+1}) \geq M(d\lambda^m)^{k+1} \quad (2.13)$$

and this remains true for all  $k \in \mathbb{N}$ .

Let  $z = \lim_{k \rightarrow \infty} x_k$ . Since  $t\lambda \leq 1/3$ , we have  $|z - x_k| \leq \sum_{i=k}^{\infty} (t\lambda)^i < 2(t\lambda)^k$ . Thus  $B_k \subset B(z, 3(t\lambda)^k)$  for all  $k \in \mathbb{N}$ . If  $(t\lambda)^{k+1} \leq r' < (t\lambda)^k$ , then  $3r' < (t\lambda)^{k-1}$ , and hence, using (2.13), (2.2), and (2.6), we get

$$\begin{aligned}
 \frac{\mu(B(z, 3r'))}{h(6r')} &\geq \frac{\mu(B_{k+1})}{h(2(t\lambda)^{k-1})} > \frac{M d^{k+1} \lambda^{m(k+1)}}{h(2(t\lambda)^{k-1})} \\
 &= M d^2 \lambda^{2m} (d \lambda^{m-s} t^{-s})^{k-1} \frac{(t\lambda)^{s(k-1)}}{h(2(t\lambda)^{k-1})} \\
 &\geq \frac{M d^2 \lambda^{2m} 2^{(s-m)(k-1)}}{h(2)} \rightarrow \infty
 \end{aligned}$$

as  $r' \downarrow 0$ . This implies  $\underline{D}_h(\mu, z) = \infty$ , giving a contradiction since  $z \in F$ . This completes the proof in the case  $m \geq 1$ .

When  $m = 0$ , the proof is actually easier since we do not need to consider the slices  $V_i^y$ . We argue by contradiction that there is a compact set  $F$  with

$\mu(F) > 0$  so that  $\underline{D}_h(\mu, x) < M$  and (2.7) is satisfied for all  $x \in F$  (the cones  $X(x, r, V^i, \alpha)$  are replaced by  $B(x, r)$ ,  $l = 1$ , and the infimum is only over all  $\theta \in S^{n-1}$ ). Then we define  $B_0$  such that  $\mu(F \cap B_0) \geq Mh(\text{diam}(B_0))$  and for  $k \geq 0$  we choose the balls  $B(y_j, (t\lambda)^{k+1} \text{diam}(B_0)/2)$  for  $y_1, \dots, y_q \in F \cap B_k$  as above. Finally, we use Lemma 2.1 to get a lower bound for  $\mu(F \cap B_{k+1})$  yielding a point  $z \in F$  for which  $\underline{D}_h(\mu, z) = \infty$ .  $\square$

Let us now consider the most important special cases of Theorem 2.4. Let  $h_s(r) = r^s$  as  $r \geq 0$ . As noted in the introduction, Theorem 2.4 is a generalization of (1.4). This follows from the well known fact that

$$2^{-s} \leq \overline{D}_{h_s}(\mathcal{H}^s|_A, x) \leq 1$$

for  $\mathcal{H}^s$ -almost all  $x \in A$  provided that  $A \subset \mathbb{R}^n$  with  $0 < \mathcal{H}^s(A) < \infty$ . The most important improvement in Theorem 2.4 compared to (1.4) is related to the  $s$ -dimensional packing measure,  $\mathcal{P}^s$ . See [14, §5.10] for the definition. If  $A \subset \mathbb{R}^n$  with  $0 < \mathcal{P}^s(A) < \infty$  then

$$\underline{D}_{h_s}(\mathcal{P}^s|_A, x) = 1$$

for  $\mathcal{P}^s$ -almost all  $x \in A$ , see [14, Theorem 6.10]. Thus we get the following corollary:

**Corollary 2.5.** *Suppose  $0 \leq m < s \leq n$  and  $0 < \alpha, \eta \leq 1$ . Then there is a constant  $c = c(n, m, s, \alpha, \eta) > 0$  such that*

$$\begin{aligned} \limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mathcal{P}^s(A \cap X(x, r, V, \alpha) \setminus H(x, \theta, \eta))}{(2r)^s} & \quad (2.14) \\ & \geq c \overline{D}_{h_s}(\mathcal{P}^s|_A, x) \geq c \end{aligned}$$

for  $\mathcal{P}^s$ -almost every  $x \in A$  whenever  $A \subset \mathbb{R}^n$  with  $0 < \mathcal{P}^s(A) < \infty$ .

It is remarkable to note that the upper density  $\overline{D}_{h_s}(\mathcal{P}^s|_A, x)$  may be infinity almost everywhere on the set  $A$ . In this case Corollary 2.5 states that also the upper density (2.14) is infinity for  $\mathcal{P}^s$ -almost every  $x \in A$ .

For many fractals some other gauge function than  $h_s$  might be more useful in measuring the fractal set in a delicate manner. Denote the Hausdorff and packing measures constructed using the gauge  $h$  by  $\mathcal{H}_h$  and  $\mathcal{P}_h$ , respectively. See [14, §4.9] and [3, Definition 3.2] for the definitions. If  $A, B \subset \mathbb{R}^n$ ,  $0 < \mathcal{H}_h(A) < \infty$ ,  $0 < \mathcal{P}_h(B) < \infty$ ,  $\mu = \mathcal{H}_h|_A$ , and  $\nu = \mathcal{P}_h|_B$ , then  $\liminf_{r \downarrow 0} h(r)/h(2r) \leq \overline{D}_h(\mu, x) \leq 1$  for  $\mu$ -almost every  $x \in \mathbb{R}^n$  and  $\underline{D}_h(\nu, x) = 1$  for  $\nu$ -almost every  $x \in \mathbb{R}^n$ . Thus Theorem 2.4 may be applied to measures  $\mu$  and  $\nu$  provided that  $h$  satisfies any of the conditions (1)–(3) of Lemma 2.3. These conditions hold for functions such as  $h(r) = r^s / \log(1/r)$  or  $h(r) = r^s \log(1/r)$ ,  $s > m$ . However, some gauge functions such as  $h(r) = r^m / \log(1/r)$  fail to satisfy them although  $\lim_{r \downarrow 0} h(r)/r^m = 0$ . For this gauge, Theorem 2.4 is not even true as will be shown in Proposition 3.3.

## 3. POROSITY AND CONICAL DENSITIES

In this section we discuss relations between conical upper density theorems and porosity of measures. Our application concerns the following definition of lower porosity of measures. Let  $k$  and  $n$  be integers with  $1 \leq k \leq n$ . For all locally finite Borel measures  $\mu$  in  $\mathbb{R}^n$ ,  $x \in \mathbb{R}^n$ ,  $r > 0$ , and  $\varepsilon > 0$ , we set

$$\begin{aligned} \text{por}_k(\mu, x, r, \varepsilon) = \sup\{\varrho : \text{there are distinct } z_1, \dots, z_k \in \mathbb{R}^n \setminus \{x\} \text{ such that} \\ B(z_i, \varrho r) \subset B(x, r) \text{ and } \mu(B(z_i, \varrho r)) \leq \varepsilon \mu(B(x, r)) \\ \text{for every } i \text{ and } (z_i - x) \cdot (z_j - x) = 0 \text{ if } j \neq i\}. \end{aligned}$$

The  $k$ -porosity of  $\mu$  at a point  $x$  is defined to be

$$\text{por}_k(\mu, x) = \lim_{\varepsilon \downarrow 0} \liminf_{r \downarrow 0} \text{por}_k(\mu, x, r, \varepsilon),$$

When  $k = 1$ , our definition of  $\text{por}_1$  agrees with the lower porosity of measures introduced by Eckmann, Järvenpää and Järvenpää in [4]. When  $k > 1$ , our definition of  $k$ -porosity is a natural generalization of the  $k$ -porosity of sets studied in [10] and [11]. For a motivation, examples, and more information on dimension of lower porous sets and measures, consult [9] and [11]. It is possible that  $\text{por}_k(\mu, x) > 1/2$  in a single point but  $\text{por}_k(\mu, x) \leq 1/2$  for almost every  $x$  for any Borel measure  $\mu$ , see [4, p. 4].

If  $0 < \alpha < 1$  and  $m, n \in \mathbb{N}$ , we denote

$$\begin{aligned} V &= \{x \in \mathbb{R}^n : x_i = 0 \text{ for all } i = 1, \dots, n - m\}, \\ C &= \{x \in \mathbb{R}^n : x_i > 0 \text{ for } i = 1, \dots, n - m\}, \end{aligned}$$

and  $\theta = (n - m)^{-1/2} \sum_{i=1}^{n-m} e_i \in S^{n-1}$  and define

$$\eta(\alpha, m, n) = \sup\{\eta \geq 0 : C \cap X(0, V, \alpha) \subset H(0, \theta, \eta)\},$$

where  $X(x, V, \alpha) = X(x, V, \infty, \alpha) = \{y \in \mathbb{R}^n : \text{dist}(y - x, V) < \alpha|y - x|\}$ . Moreover, if  $0 < \eta < \eta(\alpha, n, m)$ , we put  $x_0 = \sum_{i=1}^{n-m} e_i \in \mathbb{R}^n$  and

$$\tilde{c}(\eta) = \tilde{c}(\eta, \alpha, n, m) = \inf\{r > 0 : C \cap X(0, V, \alpha) \setminus B(0, r) \subset H(x_0, \theta, \eta)\}. \quad (3.1)$$

By simple geometric inspections, one checks that  $\eta > 0$  and  $\tilde{c} < \infty$  though the exact values may be hard to compute.

**Theorem 3.1.** *Let  $h$  satisfy the doubling condition*

$$\limsup_{r \downarrow 0} h(2r)/h(r) < \infty \quad (3.2)$$

*and suppose further that*

$$h(\varepsilon r)/h(r) \xrightarrow{\varepsilon \downarrow 0} 0 \quad (3.3)$$

*uniformly for all  $0 < r < r_0$ . Assume that  $0 \leq m < n$ ,  $0 < \alpha < 1$ , and  $0 < \eta < \eta(\alpha, n, m)$ . Let  $\mu$  be a Borel measure on  $\mathbb{R}^n$  with  $0 < \overline{D}_h(\mu, x) < \infty$  for*

$\mu$ -almost all  $x \in \mathbb{R}^n$  and suppose there is  $c > 0$  such that

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mu(X(x, r, V, \alpha) \setminus H(x, \theta, \eta))}{h(2r)} \geq c\overline{D}_h(\mu, x) \quad (3.4)$$

for  $\mu$ -almost every  $x \in \mathbb{R}^n$ . Then  $\text{por}_{n-m}(\mu, x) \leq 1/2 - c'$  for  $\mu$ -almost every  $x$ , where  $c' > 0$  is a constant depending only on  $n, m, \alpha, \eta, c$ , and  $h$ .

*Proof.* The argument is purely geometric though a bit technical. The idea is similar to those in the proofs of [11, Theorem 3.2] and [14, Theorem 11.14].

Denote  $k = n - m$  and suppose that  $\text{por}_k(\mu, x) > \varrho > \sqrt{2} - 1$  in a measurable set  $A \subset \mathbb{R}^n$  with  $\mu(A) > 0$ . Let  $t = (1 - 2\varrho)^{-1/2}$  and  $\delta = t(1 - \varrho - (\varrho^2 + 2\varrho - 1)^{1/2})$ . Then

$$H(x + \delta r\theta, \theta) \cap B(x, r) \subset B(z, \varrho tr) \quad (3.5)$$

whenever  $\theta \in S^{n-1}$  and  $B(z, \varrho tr) \subset B(x, tr)$ , see [11, Lemma 3.1]. Here  $H(x, \theta) = H(x, \theta, 0)$ . Since  $\delta = \delta(\varrho) \downarrow 0$  as  $\varrho \uparrow 1/2$ , it suffices to find a positive lower bound for  $\delta$  depending only on  $c, h, \alpha, \eta, n$ , and  $m$ .

By (3.4), we may find  $x \in A$  for which  $0 < \overline{D}_h(\mu, x) = M < \infty$  and

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, k)}} \frac{\mu(X(x, r, V, \alpha) \setminus H(x, \theta, \eta))}{h(2r)} \geq cM.$$

Using (3.2), we may choose  $\varepsilon > 0$  so small that

$$\varepsilon h(2tr) < h(2\tilde{c}\delta r) \quad (3.6)$$

for all  $0 < r < r_0$ , where  $\tilde{c} = \tilde{c}(\eta)$  is as in (3.1). Next choose  $0 < r_1 < r_0$  such that

$$\text{por}_k(\mu, x, r, \varepsilon/k) > \varrho \quad \text{and} \quad \mu(B(x, r)) < 2Mh(2r) \quad (3.7)$$

for all  $0 < r < r_1$ . Now we take  $0 < r < \min\{r_1/t, r_1/(2\tilde{c}\delta)\}$  such that

$$\inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, k)}} \mu(X(x, r, V, \alpha) \setminus H(x, \theta, \eta)) > cMh(2r)/2. \quad (3.8)$$

Using (3.7), we find  $z_1, \dots, z_k \in B(x, (1 - \varrho)r) \setminus \{x\}$  with  $(z_i - x) \cdot (z_j - x) = 0$  as  $i \neq j$  and  $\mu(B(z_i, \varrho tr)) \leq \varepsilon\mu(B(x, tr))/k$  for all  $i \in \{1, \dots, k\}$ . In particular,

$$\mu\left(\bigcup_{i=1}^k B(z_i, \varrho tr)\right) \leq \varepsilon\mu(B(x, tr)) \leq 2\varepsilon Mh(2tr). \quad (3.9)$$

Let  $\theta_i = (z_i - x)/|z_i - x|$  for  $i \in \{1, \dots, k\}$ . Applying (3.5), we see that  $H(x + \delta r\theta_i, \theta_i) \cap B(x, r) \subset B(z_i, \varrho tr)$  for every  $i$ . If  $V \in G(n, k)$  is the  $k$ -plane spanned

by the vectors  $\theta_1, \dots, \theta_k$  and  $\theta = -k^{1/2} \sum_{i=1}^k \theta_i$  then we conclude that

$$\begin{aligned} & (X(x, r, V, \alpha) \setminus H(x, \theta, \eta)) \setminus \bigcup_{i=1}^k B(z_i, \varrho tr) \\ & \subset (X(x, r, V, \alpha) \setminus H(x, \theta, \eta)) \setminus \bigcup_{i=1}^k H(x + \delta r \theta_i, \theta_i) \subset B(x, \tilde{c} \delta r) \end{aligned}$$

using the definition of  $\tilde{c}$  for the last inclusion.

Using (3.8), the above inclusion, the latter condition of (3.7), (3.9), and (3.6), we conclude that

$$\begin{aligned} cMh(2r)/2 & < \mu(X(x, r, V, \alpha) \setminus H(x, \theta, \eta)) \\ & \leq 2Mh(2\tilde{c}\delta r) + 2\varepsilon Mh(2tr) \leq 4Mh(2\tilde{c}\delta r). \end{aligned}$$

This reduces to  $h(2\tilde{c}\delta r)/h(2r) > c/8$  and thus by (3.3), we must have  $\delta > \delta_0$  for  $\delta_0 > 0$  depending only on  $c, h, n, \alpha$ , and  $\eta$ .  $\square$

As an immediate consequence of Theorems 2.4 and 3.1, we get the following corollary for the  $k$ -porosity of Hausdorff type measures:

**Corollary 3.2.** *Suppose  $h$  and  $\mu$  satisfy the assumptions of Theorem 2.4, (3.2), and  $0 < \overline{D}_h(\mu, x) < \infty$  almost everywhere. Then  $\text{por}_{n-m}(\mu, x) < 1/2 - c$ , where  $c > 0$  is a constant depending only on  $m, n, s$ , and  $\varepsilon_0$ .*

When  $m = n - 1$  and  $h = h_s$ , Corollary 3.2 is a special case of [9, Corollary 2.9].

We do not know if it is possible to find weaker conditions for  $h$  than the ones in Lemma 2.3 under which Theorem 2.4 holds. However, we may use Theorem 3.1 to rule out some possible generalizations.

**Proposition 3.3.** *Suppose  $h$  satisfies (3.2) and (3.3). Suppose further that there is an integer  $1 \leq m \leq n - 1$  and a decreasing sequence  $(r_j)$  for which  $h(r_{j+1}) \geq 2^{m-n}(r_{j+1}/r_j)^m h(r_j)$  and  $r_j/r_{j+1} \rightarrow \infty$  as  $j \rightarrow \infty$ . Then there is a measure  $\mu$  on  $\mathbb{R}^n$  for which  $0 < \overline{D}_h(\mu, x) < \infty$  for  $\mu$ -almost all  $x \in \mathbb{R}^n$  and*

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mu(X(x, r, V, \alpha) \setminus H(x, \theta, \eta))}{h(2r)} = 0 \quad (3.10)$$

for  $\mu$ -almost every  $x \in \mathbb{R}^n$  and for all  $0 < \alpha < 1$  and  $0 < \eta < \eta(\alpha)$ . Here  $\eta(\alpha) = \eta(\alpha, m, n)$  is as in Theorem 3.1.

*Proof.* We may assume that  $r_{j+1} < r_j/2$  for all  $j$ . Let  $\tilde{h}(r) = r^{-m}h(r)$ . Then  $\tilde{h}(r_{j+1}) \geq 2^{m-n}\tilde{h}(r_j)$  for all  $j \in \mathbb{N}$ . Let  $Q \subset \mathbb{R}^{n-m}$  be a closed cube with side-length  $r_0$  and let  $Q_{1,1}, \dots, Q_{1,2^{n-m}} \subset I$  be the closed cubes located at the corners of  $Q$  with side-length  $r_1$ . In a similar manner, divide  $Q_{1,1}, Q_{1,2^{n-m}}$  into totally

$2^{2(n-m)}$  subcubes of side-length  $r_2$ , say  $Q_{2,1}, \dots, Q_{2,2^{2(n-m)}}$ . Continuing in this manner, we define a Cantor type set  $C = \bigcap_{j \in \mathbb{N}} \bigcup_{i=1}^{2^{j(n-m)}} Q_{j,i} \subset \mathbb{R}^{n-m}$ . Since arbitrary covers  $\{E_k\}_k$  of  $C$  are reduced to finite covers of the sets  $Q_{j,i}$ , so that  $\sum_k \tilde{h}(\text{diam}(E_k)) \geq c \sum_i \tilde{h}(\text{diam}(Q_{j,i}))$  for a constant  $c = c(n, m) > 0$ , we easily obtain  $\mathcal{H}_{\tilde{h}}(C) > 0$ . If  $A = C \times [0, 1]^m$  then, by applying the calculations done in [14, Theorem 7.7], we have  $\mathcal{H}_h(A) > 0$ . Now we may find a compact  $F \subset A$  with  $0 < \mathcal{H}_h(F) < \infty$ , see [8]. For  $\mu = \mathcal{H}_h|_F$ , we then have  $0 < \overline{D}_h(\mu, x) < \infty$  for  $\mu$ -almost all  $x \in \mathbb{R}^n$ . Since  $r_j/r_{j+1} \rightarrow \infty$ , it is easy to see that  $\text{por}_{n-m}(\mu, x) = 1/2$  for  $\mu$ -almost every  $x \in F$ . By Theorem 3.1, this implies (3.10) for  $\mu$ -almost all  $x$  whenever  $0 < \alpha < 1$  and  $0 < \eta < \eta(\alpha)$ .  $\square$

*Remark 3.4.* Inspecting the proofs of Proposition 3.3 and Theorem 3.1, it is easily seen that  $V$  may be fixed in (3.10).

Let us compare the assumptions of the above proposition with the assumptions of Theorem 2.4. Recall, by Lemma 2.3, that in Theorem 2.4 our assumption for  $h$  is: There is  $0 < c < 1$  such that  $\limsup_{r \downarrow 0} h(cr)/h(r) < c^m$ . On the other hand, if

$$\liminf_{r \downarrow 0} h(cr)/h(r) \geq c^m$$

for all  $0 < c < 1$  then the assumptions of Proposition 3.3 are clearly satisfied. This shows that Theorem 2.4 does not hold for gauge functions such as  $h(r) = r^m/\log(1/r)$  when  $m > 0$ .

#### 4. OPEN PROBLEMS

We discuss below some of the questions raised by Theorem 2.4.

*Question 4.1.* Most measures are so unevenly distributed that there are no functions that could be used to approximate the measure in small balls. For these measures it is natural to study upper densities such as

$$\limsup_{r \downarrow 0} \frac{\mu(X(x, r, V, \alpha))}{\mu(B(x, r))}.$$

In order to bound these densities from below, we need to guarantee that the measure  $\mu$  is not concentrated in too small regions. One way to do this is to impose bounds on the dimension of the measure. We pose the following open problem. It is stated here in its simplest form though natural generalizations arise by analogy with (1.1)–(1.4): Suppose that  $\mu$  is a Borel measure on  $\mathbb{R}^n$  whose packing dimension,  $\dim_p(\mu)$ , equals  $s$  (see [6, §10]). If  $0 < \alpha < 1$ ,  $m \in \mathbb{N}$  with  $m < s$ , and  $V \in G(n, n-m)$ , is it true that

$$\limsup_{r \downarrow 0} \frac{\mu(X(x, r, V, \alpha))}{\mu(B(x, r))} \geq c \tag{4.1}$$

for  $\mu$ -almost every  $x \in \mathbb{R}^n$ , where  $c > 0$  depends only on  $n, m, s$ , and  $\alpha$ ? If  $\mu$  satisfies almost everywhere the doubling condition

$$\limsup_{r \downarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < d < \infty, \quad (4.2)$$

the answer is known to be yes: In fact, for any Borel measure  $\mu$  that satisfies (4.2) and is purely  $m$ -unrectifiable in the sense that  $\mu(E) = 0$  for all  $m$ -rectifiable sets  $E \subset \mathbb{R}^n$ , the claim (4.1) holds with a constant  $c = c(d, \alpha) > 0$ . This follows by inspecting the proof of [14, Lemma 15.14].

*Question 4.2.* A related question concerning purely unrectifiable sets is the following: Suppose that  $E \subset \mathbb{R}^n$  is purely  $m$ -unrectifiable,  $0 < \mathcal{H}^m(E) < \infty$ , and  $\mu = \mathcal{H}^m|_E$ . Is

$$\limsup_{r \downarrow 0} \inf_{V \in G(n, n-m)} \frac{\mu(X(x, r, V, \alpha))}{(2r)^m} \geq c(n, m, \alpha) > 0$$

for  $\mu$ -almost every  $x$ ? This would be the analogy of (1.3) for purely unrectifiable sets. The analogy of (1.1) in this case is well known. On the other hand, the analogy of (1.4) does not hold under these assumptions, even if we fix  $V$ . A set of Besicovitch [2, p. 327] serves as a counterexample.

*Question 4.3.* Inspecting Proposition 3.3, one recognizes that there are no gauge functions satisfying its assumptions when  $m = 0$ . This leads to ask if Theorem 2.4 for  $m = 0$  is true for all gauge functions. That is, whether for all  $0 < \eta < 1$  there is  $c = c(n, \eta) > 0$  such that

$$\limsup_{r \downarrow 0} \inf_{\theta \in S^{n-1}} \frac{\mu(B(x, r) \setminus H(x, \theta, \eta))}{h(2r)} \geq c \overline{D}_h(\mu, x)$$

for all gauge functions  $h$ , all Borel measures  $\mu$ , and  $\mu$ -almost every  $x \in \mathbb{R}^n$ ? When  $n = 1$ , this is known and reads

$$\limsup_{r \downarrow 0} \frac{\min\{\mu([x, x+r]), \mu([x-r, x])\}}{h(2r)} \geq \overline{D}_h(\mu, x)/4$$

for  $\mu$ -almost all  $x \in \mathbb{R}$ . This follows from the proof of [19, Theorem 3.1].

*Question 4.4.* When  $k > 1$ , we do not know if Theorem 3.1 holds for packing type measures, that is, for measures with  $0 < \underline{D}_h(\mu, x) < \infty$ . When  $k = 1$ , a more general result is obtained in a forthcoming paper [1].

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